SCHOOL MATHEMATICS COMPETITION for 2018

The Hamann School Mathematics Competition for
Society of Petroleum Engineers (SPE) Prizes

SAMPLE QUESTIONS

The following sample competition problems involve extensions to the basic mathematics concepts and skills that you already have developed. The emphasis is on algebra and geometry as an understanding and appreciation of their fundamentals is important in the development of many future tertiary studies. We have tried to set the problems in a non-standard form in order to allow you to experience the creative and diverse nature of mathematics. We hope that you will find them interesting and solvable without being tedious or impossible.

Answers and solutions are available from the MASA Office

STARTER SET

1. List all the prime numbers between 1 and 1000 that have two 3's in them.

2. In a group of dogs and people the number of legs was 28 more than the number of heads. How many people were there?

3. Three footballers were weighed two at a time. These weights were 155 kg, 152 kg and 147 kg. How much did the heaviest footballer weigh?

4. In the triangle ABC the point D lies on AB such that AD= BD = CD. If the angle BAC is 64 degrees, what is the size of the angle BCD?

5. The circumference of a circle with centre O is divided into 12 equal arcs marked A, B, C, …, L. What is the sum of the angles OAE and OGI?

6. Stacey has a week to read a book for a school assignment. She reads an average of 36 pages on the first 3 days and an average of 44 pages for the next three days. She finishes the book by reading 19 pages on the last day. What was her average pages read each day over the week?

7. The 7 digit number 74A52B1 and 326AB4C are each multiples of 3. What is the value of C?

8. Jake walks 2 km to school each day by the same route. He leaves at the same time each and walks at a steady 6 kph and arrives just in time for school. Today he was distracted and walked the first km at only a steady 4 kph. At what speed, in kph, must Jake walk the last 1 km in order to arrive just as school is beginning?

9. AB is a two digit number (A and B are both digits) such that the product of the digits plus the sum of the digits is equal to the number. What is the number?

10. Of the 500 balls in a large bag 80% are red and the rest blue. How many of the red balls must be removed so that 75% of the balls are red?

11. The school band is selling pizza to raise money for a music festival. The large pizza's sell for $10.50 each and the smaller pizza's sell for $8 each. If the total sales were $945 and the ratio of large pizza's sold to the small pizza's sold was 2: 3, how many large pizza's were sold?

12. A rectangular swimming pool is surrounded by a concrete path which is 2 metres wide. The area of the path is 116 m^2 and the length and breadth of the pool are both square numbers. What is the perimeter of the pool?

13. Every day Jill climbs a flight of 6 stairs. She can take the stairs 1, 2 or 3 steps at a time, for example, Jill can climb 3, then 1 then 2 steps. In how many ways can Jill climb the stairs?

14. In a room $\frac{2}{5}$ of the people are wearing gloves and $\frac{3}{4}$ of the people are wearing hats. What is the smallest number of people in the room wearing both gloves and a hat?

15. A, B, C, and D are four digits selected from the numbers 1,2,3,4,5,6,7,8, 9. What is the smallest value that $\frac{A}{B} + \frac{C}{D}$ can have?

16. The sum of the 3-digit numbers 35X and 4Y7 is N, where X and Y are different digits.
    i) What is the smallest value that N can have?
    ii) What is the largest value that N can have?
    iii) If N = 851 what are the values of X and Y?

17. When a 2-digit number is multiplied by 1111, the answer can be a 5-digit number or a 6-digit number.
    i) What is the smallest 5-digit number you can get?
    ii) What is the largest 6-digit number you can get?
    iii) What is the smallest 6-digit number you can get?
    iv) What is the largest 5-digit number you can get?

18. Suppose there are two numbers A and B where B is the number obtained by writing the number A in reverse, Thus for example if A = 856 then B = 658. If the product of A and B is 78445 what are the numbers A and B?

19. There are some numbers, not necessarily distinct, each of them between 13 and 19, such that their product is 2940. What are these numbers?

20. The average of three positive numbers is 28. When two additional numbers are added, the average of all five numbers is 34. What is the average of the last two numbers added?

21. The two digit numbers from 1 to 30 are written consecutively and the number

N = 12345678910111213…2627282930

is formed.

i) How many digits in this number?

ii) What is the sum of all the digits in this number?

iii) If we delete 30 digits from this number, without altering the position of the numbers remaining, what is the smallest number you can be left with?

22. In the addition sum

$$\begin{array}{r} a\ b\ c \\ +\ \ d\ e\ f \\ \hline 1\ 3\ 6\ 2 \end{array}$$

with a. b. c, d, e, f distinct digits, what is the value of a + b + c + d + e + f ?

23, A box contains a total of 400 marbles that come in five colours; blue, green, red, yellow and orange. The ratio of blue: green: red is 1 :2 : 4 and the ratio green : yellow : orange = 1 : 3 : 6 .
What is the smallest number of marbles that must be drawn from the box, and not replaced, to ensure that at least 50 marbles of one colour have been drawn ?

24. Adam and Eve want to buy the same book. Adam has ¾ of the money needed to buy the book and Eve has ½ of the money needed to buy the book. If the book was $3 cheaper, then together they would have exactly enough money to buy 2 copies of the book?
What is the original price of the book?

25. Given that a, b and c are digits such that $a + \frac{1}{b+\frac{1}{c}} = \frac{11}{3}$ what are the values of a, b and c?

JUNIOR PROBLEMS

1. Find all the 3-digit multiples of 7 for which the sum of the digits is also a multiple of 7.

2. Joe walks up10 steps going up either 1 or 2 steps at a time. There is a snake on the 6th step so he cannot stop on this step. In how many different ways can Joe reach the top step?

3. How many pairs (x,y), where x and y are positive integers , satisfy the condition that 2x + 7y = 100?

4. By how much does $\frac{5}{7}$ of $9\frac{1}{3}$ exceed $\frac{3}{11}$ of $4\frac{2}{5}$?

5. N is a whole number and S(N) is the sum of the digits of N.
Thus, for example, if N = 372 then S(N) = 12
If N = 1982 then S(N) = 20
Find all the numbers for which N + S(N) = 2010.

6. 90% of the members in a certain gym club are girls and there are 3 boys in the class. If there are less than 100 in the club, how many members could there be in the club?

7. Amanda has received a 5% raise. Now she earns $1200 more than her friend Lisa. Before Amanda's salary raise, Lisa's salary was 1% higher than Amanda's. What is Lisa's salary?

8. P is a convex polyhedron with 60 edges and 36 faces. 24 of the faces are triangular and 12 are quadrilaterals. A space diagonal is a line segment connecting two vertices which do not belong to the same face.
   a. How many vertices does the polyhedron have?
   b. How many space diagonals does the polyhedron have?

9. i) How many 4-digit palindromic numbers start with the digit 7?
   ii) How many palindromic numbers lie between 2000 and 20,000?

10. The sum of the 3-digit numbers 35x and 4y7, where x and y are digits, is N.
   i) What is the smallest value that N can take?
   ii) What is the largest value that N can take?
   iii) If N is divisible by 36 what is the value (or values) that N can have?

11. Consider the sequence of numbers
1 , 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, 1, …
If this pattern continues what will be
   i) the 2018th number in the sequence,
   ii) the sum of the first 2018 terms in the sequence?

12. N is a digit (0,1,2,3,4,5,6,7,8,or 9)
If N3 x 6528 = 3N x 8256 what is the value of N?
(N3 and 3N are two digit numbers)

13. The total cost of 1 alpha, 3 betas and 7 gammas is $14 while the cost of 1 alpha, 4 betas and 10 gammas is $17.

What would be the cost of 3 alphas and 2 betas?

14. A bag of apples cost $4, a box of oranges cost $3 and a box of lemons cost $2 (they are going out of business with these prices). Ross buys 8 boxes of fruit at a total cost of $23. What is the largest possible number of boxes of apples that Ross can purchase?

15. During a particular morning, a signal light goes on at exactly 9.00 am. Thereafter it goes off and on at equal intervals, each lasting a whole number of minutes. Later that morning, it is observed that the light is off at 9.09 am, is on at 9.17 am and on at 9.58.

For what length of time does the light stay on?

16. Each of the letters represent a different digit in the multiplication

$$ME \times HE = XXX$$

Find the value of each letter.

17. The average number of degrees in the angles of a convex polygon is 150. How many sides does this polygon have?

18. The numbers a, b, c are the digits of the three digit number (abc) which satisfy the equation $49a + 7b + c = 286$.

What is the three digit number $190a + 10b + c$?

19. How many 5 digit positive numbers have the product of their digits equal to 2000?

20. 13 points are equally spaced on a circle. Five of these vertices are coloured red. Prove that you can always find 3 of these red points that they are the vertices of an isosceles triangle?

21. Consider the 25 odd numbers 1, 3, 5, 7, … , 47, 49. If we choose any 14 of these odd numbers explain why there will always be two of them whose sum is 50.

22. The sum of 16 distinct integers is less than 100 and the sum of their squares is less than 1000. Explain that none of the 16 integers is greater than 25.

23. Four dice, coloured red, blue, green and black are rolled. In how many ways can the product of the numbers rolled equal 36?
(Note : a red 4 is considered different to a blue 4 and so on.)

24. Find the smallest positive integer by which 139392 must be multiplied by in order that the answer is i) a perfect square, ii) a perfect cube?

25. From a group of boys and girls, 15 girls leave. There are then two boys for each girl. After this 45 boys leave and there are then 5 girls for each boy. How many people in the original group?

26. Wayne loves a challenge. He can ascend a descending escalator in 40 seconds by making 70 steps. By taking 40 steps, he can descend the same escalator in 20 seconds.

How many steps are exposed on the escalator?

27. A straight rod is divided into segments of equal length by 9 marks. If the rod is cut at two of these marks, the three resulting pieces may form the sides of a triangle. List the side lengths of all the possible triangles that may be formed. (Note : there are 9 MARKS on the rod.)

28. How many numbers divisible by n are there between

a) n and n^2 b) n and n^3 ?

29. Find all the numbers that are perfect squares and perfect cubes that lie between 1,000,000 and 10,000,000.

30. A man lost in the desert took refuge at an oasis. After a week with no help arriving he decides to walk 80 kilometres out of the desert. How long will it take him if he can walk 20 km per day, but can only carry enough food and water for 3 days?

(He has ample food and water stocks so he can stock food and water at places he reaches at the end of a day of walking.)

31. Four girls bought a windsurfer for \$300. The first girl paid ½ of the sum of the total amount paid by the other girls, the second paid $\frac{1}{3}$ of the total amount paid by the other girls while the third girl paid $\frac{1}{4}$ of the sum of the total amount paid by the other girls.

How much did each of the girls pay?

32. Consider the sequence of numbers

1, 2, 4, 5, 7 ,9, 10, 12, 14 , 16, 17, 19, 21, 23, 25, 26, …

consisting of the first odd number, then the next two even integers, the next three odd integers, the next four even integers, and so on. What is the 2018th. term in this sequence?

33. The letters of the word DECIMAL are arranged in a triangular array as shown below. In how many ways can the word DECIMAL be selected from this array if you must start from the top right D and you must move either down or to the right.

D E C I M A L
E C I M A L
C I M A L
I M A L
M A L
A L
L

INTERMEDIATE PROBLEMS

1. List all the prime numbers between 1 and 1000 that have two 7's in them

2. In triangle ABC, D is a point on AB such that AD = BD = CD If the angle BAC = 52 degrees, what is the size of the angle BCD?

3. How many integers between 1 and 1000 are divisible by 4 and 6 but are not divisible by 26?

4. What is the value of $8^{4/3}(2^{-3} - 9^{-1/2})$

5. The average of a set of 50 numbers is 45 and the average of another set of n numbers is 65 If the combined average of the two sets is 60 what is the value of n?

6. There are 30 students in a room 60% of whom are boys. How many girls must enter the room so that 40% of the total number of people in the room are boys?

7. The diagonals of a quadrilateral are perpendicular. If the lengths of three of the sides are 2, 3, and 4, what are the possibilities for the length of the fourth side?

8. The sum of the squares of three prime numbers is 302. What is the sum of these three prime numbers?

9. A circular yard has a radius of 12 metres. A small horse is tethered to a point on the edge of the yard by a lead of length 12 metres. What area of the yard can the horse graze on?
   (Regards the horse as a point)

10. If Nicole's age is increased by 56 it is a square number. If her age is decreased by 56, the answer is also a square number.
    How old is Nicole?

12. For what values of n is $\frac{20-n}{14-n}$ a positive integer?

13. A,B,C,D are the vertices of a square piece of paper. A is folded onto C and then B is folded onto D. The area of the resulting figure is $9cm^2$
    What is the perimeter of the square ABCD?

14. There are positive integers with the following properties
    i) the sum of the squares of the digits is 50, and
    ii) each digit is larger than the one on the left.
    What is the smallest number with this property?

15. In the parallelogram ABCD, DE is the altitude to the base AB and DF is the altitude to the base BC. If DC = 12, EB = 4 and DE = 6, what is the length of DF?

16. Jill walks up 12 steps going up either 1 or 2 steps at a each stride. There is a snake on the 7th step so she cannot stop there. In how many different ways can Jill reach the top step?

17. A chemist has 100 cc of a liquid which contains 20% alcohol and 80% water. She adds more alcohol to make a solution with 331/3% alcohol. How much water must she now add in order to return it to a 20% alcohol solution?

18. How many 3-letter sequences can be made using the letters in the word "ADELAIDE" ? (For example, "LAA" is acceptable but "AAA" is not.)

19. N is a 6-digit number formed using an arrangement of the digits 1, 2, 3, 3, 4, and 5.
What is the smallest N that is divisible by 264?

20. Peta solves the equation ax – b = c for x and Rosa solves the equation bx – c = a for x. If they both get the same answer for x and a, b, and c are both distinct and non-zero, what is the relation between a, b and c?

21. The lengths of the equal sides of an isosceles triangle are (x + 1) and the length of the third side is (3x - 2). Find all the possible values of x. (The triangle should not be degenerate and your answer need not be an integer and could be an inequality.)

22. What is the number of positive integers n for which there is a triangle with 3 acute angles and the side lengths are 10, 24 and n?

23. The numbers 1, 2, 3, 4, …., 100 are divided into two groups A and B. You take a number a from A and a number b from B and you add them to make the number (a + b).
What is the largest value of (a + b) that you can make?
What is the largest number of different answers that you can make for (a + b)?

24. ABCD is a rhombus. O is a point on diagonal AC such that AD = AO = x and OD = OB = OC. Find the value of x.

25. N = 9,999,999 How many 9's in the number N^2 ?

26. Let S = { 15, 24, 30, 40, 50, 60, 80 } When one number is removed, the product of the remaining numbers is a perfect cube.
What number is this?

27. Alex puts 20 green marbles in a hat, Lynn puts 16 purple marbles in the hat and Josh places k blue marbles into the hat. If a marble is randomly selected from the hat, the probability that it will be blue is 1/5 . What is the value of k?

28. Find all the positive prime numbers p such that $p^{2018} + p^{2019}$ is a perfect square.

29. Given that a, b and c are prime numbers, solve the simultaneous equations

$$a(b + c) = 234 \qquad \text{and} \qquad b(a + c) = 220$$

30. When we divide a 3-digit number N by 28 the remainder is 1. When we divide N by 29 we get a remainder of 1 also.
What values can N take?

31. Without using a calculator, determine which of the two numbers is the largest

$$31^{19} \quad \text{or} \quad 17^{24} \quad ?$$

32. Evaluate the sum $[\sqrt{1}] + [\sqrt{2}] + [\sqrt{3}] + \ \dots + [\sqrt{n}\}$ for n = 100
What is the sum when n = 1000?
[n] is the greatest integer less than or equal to n

33. i) Express 25! (factorial 25) as the product of primes
ii) How many zeros at the end of 25! ?
iii) k! ends in 10 zeros. What is the value of k?

34. The point M is chosen inside the square ABCD such that the angle MAC = the angle MCD = x degrees.
Find the size of the angle ABM in terms of x.

35. Point P is 6 units from O, the centre of a circle of radius 10 units. How Many chords with integer length can be drawn through P?

36. The increasing sequence 1, 3, 4, 9, 10, 12, 13, … consist of all those positive integers which are the sums of powers of 3. Thus

$$13 = 3^2 + 3^1 + 3^0 = 9 + 3 + 1$$

a) What are the next two numbers in the sequence?
b) What is the 20^{th} term in the sequence?
c) What is the 100^{th} term in the sequence?

37. Find all the ordered triples (a, b, c) such that a, b and c are positive primes and $a^b = c - 1$

38. In a cross country run involving two school teams each school has a team of 5 runners. Each runner who finishes in the nth position contributes n points to his team's score. The team with the lowest score wins. If there are no ties among the runners, how many different winning scores are there?

39. Consider the non-decreasing sequence of positive numbers

1, 2, 2, 3, 3, 3, 4, 4, 4, 4 , 5, 5, 5, 5, 5, 6, …

in which the nth positive integer appears n times.

i) What is the 2018^{th} integer in this list?
ii) What is the sum of these 2018 numbers?

40. All students at the ABC High School and at the XYZ High school; take a certain exam. The average score for boys, for girls, and for boys and girls combined at the two schools are shown on the table below, as is the average score for boys at the two schools combined.

	ABC High	XYZ High	Combined
Boys	71	81	79
Girls	76	90	?
B's & G's	74	84	

What was the average score for the girls at the two schools combined?

41. 11 boys and n girls went picking oranges. They picked $(n^2 + 9n + 2)$ oranges in total and each child picked the same number of oranges. How many girls in the group?

42. Jack is 17 years older than Jill. If his age were written after hers , the result would be a 4 digit square number. The same statement can be made 13 years from now. What are their present ages?

43. Big Ball, a sphere of radius 20 cm , chases Tiny Ball into a corner of the room. What is the radius of Tiny ball if he just escapes being run down by Big ball?

44. Express $N = 2^{2005} + 2^{2006} + 2^{2007} + 2^{2008} + 2^{2009}$ as the product of prime factors.

45. Find the value of the non-zero digits A, B, and C for which

$$(AA)^2 = BBCC$$

where AA and BBCC are 2 and 4 digit numbers respectively.

SENIOR PROBLEMS

1. An acute angled triangle has sides of slope 1 and 7. What is the slope of the line which bisects the angle between these two lines?

2. In the system $\frac{1}{x} + \frac{1}{y} = \frac{1}{a}$; $x + y = b$; $x^2 + y^2 = c^2$ express a in terms of b and c only.

3. How many pairs of integers (x,y), $1 \le x,y \le 100$, are there such that $2x^2 + 3y^2$ is divisible by 5?

4. In a circle the chords AB and CD are perpendiculars which intersect at P. If CD = 8, AP = 4 and PB = 3 what is the diameter of the circle?

5. The lines x = -1 and x = $\sqrt{3}$ intersect the circle , centre O and radius 2. What is the area bounded between the two lines and the circle?

6. Solve for x $9^x + 2.3^{x+2} = 243$

7. A hardware store sells 3 different kinds of widgets. Green widgets sell for 25 cents each, blue widgets sell for \$1 each and red widgets sell for \$3 each. Tammy bought 75 widgets for a total cost of \$75.
   What is the greatest number of green widgets that could have been bought?

8. If f(x) = a + bx , where a and b are real, such that
   $$f(f(f(1))) = 29$$
   $$f(f(f(0))) = 2$$
   find the values of a and b.

9. Find the minimum value of the polynomial
   $$(x-1)^2 + (x-2)^2 + (x-3)^2 + \ldots + (x-2016)^2 \ ?$$

10. Solve for x $x^2 + 5x + 6 = 2\sqrt{(x^2 + 5x + 5)}$

11. In the triangle ABC, AM, BN and CP are concurrent at L where M, N and P are points on BC, AC and AB respectively. Given BM = 1, MC = 2, CN = 3, NA = 4 and AP = 5 find the length of BP.

12. What is the smallest integer N that satisfies both of the following equations where p and q are positive integers :
    $$N = p^2 + p \quad \text{and} \quad N = q^2 + q + 2016$$

13. What reduced fraction $\frac{a}{b}$ with 4 < b < 15 is closest to $\frac{3}{7}$?

14. In triangle ABC, angle A = 120, BC + AB = 21 and BC + AC = 20.
What is the length of BC?

15. Solve the system of equations $x^2 + x\sqrt[3]{xy^2} = 208$
and $y^2 + y\sqrt[3]{yx^2} = 1053$.

16. Two arithmetic sequences are multiplied together term by term to form another sequence whose first three terms are 91, 144 and 209. What is the fourth term?

17. Let p be a prime number. If p years ago, the ages of three children formed a geometric sequence with a sum of p and a common ratio of 2, what is the sum of the children's ages now?

18. If $x + \frac{1}{y} = 12$ and $y + \frac{1}{x} = \frac{3}{8}$ find all the solutions for x and y.

19. Triangle ABC, AB = 4, BC = 6 and AC = 8. Squares ABQR and BCST are drawn external to and lie in the same plane as triangle ABC. Find the length QT.

20. Consider the sequence 1, 8, 15, 22, 29, 36, 43, … where each number is obtained from the previous one by adding 7.

c. If $T_n = 7n + k$, what is the value of k?

d. $T_1 = 1$ and $T_6 = 36$ are square numbers. Find the next three square numbers.

e. Find a rule for finding the square numbers in the sequence, i.e. if T_n is a square number what is the rule for finding k in terms of n?

21. Find the smallest positive integer n such that the 11 fractions

$$\frac{3}{n+5}, \frac{4}{n+6}, \frac{5}{n+7}, \ldots, \frac{13}{n+5}$$

are all in lowest form.

22. The roots of $ax^2 + bx + c = 0$ are 6 and p. The roots of $cx^2 + bx + a = 0$ are q and r. Find the product pqr ?

23. Find the smallest prime p larger than 10, which can be expressed in the form $p = x^3 - y^3$ where x and y are positive integers.

24. Triangle ABC is right angled at B. BD lies on CB such that AD bisects the angle CAB and CB = 8 DB. Find the value of sin(angle CAD)?

25. A,B,C,D are the vertices of a square piece of paper. A is folded onto C and then B is folded onto D. The area of the resulting figure is $9cm^2$

What is the perimeter of the square ABCD?

26. In the parallelogram ABCD, DE is the altitude to the base AB and DF is the altitude to the base BC. If DC = 12, EB = 4 and DE = 6, what is the length of DF?

27. In a circle the chords AB and CD are perpendiculars which intersect at P. If CD = 8, AP = 4 and PB = 3 what is the diameter of the circle?

28. In the sum of the 7 three digit numbers below each letter represents a distinct digit, either 1, 2, 3, 4, 5, 6, 7, 8, or 9.

$$S = abc + bcd + cde + def + efg + fgh + ghi$$

What is the largest value that S can have?

29. In a plane, a set of 8 parallel lines intersect another set of n parallel lines (that go in a another direction) forming a total of 420 parallelograms, many of which overlap one another).
What is the value of n?

30. Calculate the following without the use of a calculator.
   i) $103^2 - 97^2$
   ii) $(\sqrt{7} - \sqrt{3})(\sqrt{10} + \sqrt{7})(\sqrt{7} + \sqrt{3})(\sqrt{10} - \sqrt{7})$
   iii) $(\sqrt{5} + \sqrt{3} + \sqrt{2})(\sqrt{5} + \sqrt{3} - \sqrt{2})(\sqrt{5} - \sqrt{3} + \sqrt{2})(\sqrt{5} - \sqrt{3} - \sqrt{2})$

31. Find all the numbers which are divisible by 30 and have exactly 30 different factors.
(Note : The number $p_1^{\,a} . p_2^{\,b}. \ldots$ has (a+1)(b+1) … factors .
The p's are all prime)

32. Find all the set of prime numbers a, b, c, d with a > b > c > d and

$$ab + bc + cd + da = 882$$

33. There are exactly four positive integers n such that $\frac{(n+1)^2}{n+23}$ is an integer.
Find these four values of n.

34. Prove that if p, q, and r are rational numbers and pq + qr + rp = 1 then

$$(p^2 + 1)(q^2 + 1)(r^2 + 1)$$

is the square of a rational number.

35. For every positive integer n , $h(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

Prove that for n = 2, 3, 4, …

$$n + h(1) + h(2) + h(3) + \ldots + h(n-1) = n\,h(n)$$

36. In the right triangle ABC , right angled at C, the incircle is a tangent to the hypotenuse at T.
Prove that the area of the triangle ABC is equal to AT x TB.

37. A line through a vertex A divides the triangle ABC into two isosceles triangles. It is given that one of the angles of the triangle ABC is 30^0.
Find in all possible cases the sizes of the angles of the triangle.

38. ABCD is a convex quadrilateral with AB = a, BC = b, CD = c, DA = d. If AC is perpendicular to BD prove that $a^2 + c^2 = b^2 + d^2$

CHALLENGE PROBLEMS

1. For each positive integer n let

$$a_n = \sqrt[3]{(n^2+2n+1)} + \sqrt[3]{(n^2-1)} + \sqrt[3]{(n^2-2n+1)}$$

Find the exact value of $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_{998}} + \frac{1}{a_{999}}$

2. AB is an external tangent to two circles of radii 2 cm and 5 cm with A on the small circle. AC is a diameter of the small circle and CD is a tangent to the large circle. Prove that CD = 2 cm.

3. If a + b + c = 0 prove that $2(a^4 + b^4 + c^4)$ is the square of an integer.

4. A sequence of positive integers $a_1, a_2, a_3, \ldots$ is defined as follows

$$a_{n+1} = a_n + S_n$$

where S_n is the sum of the digits of a_n.
If $100 < a_1 < 1{,}000{,}000$ show that the sequence must contain the number 63.

5. If e and f are the lengths of the diagonals of a quadrilateral of area A, prove that

$$e^2 + f^2 \geq 4A$$

When does the equality condition hold?

6. ABC is an isosceles triangle (AB = BC). D is a point on BC such that the incircle of triangle ABD and the excircle of triangle ADC tangent to DC have equal radii, Show that the radius of these circles is equal to one-quarter of the length of the altitude to the triangle ABC drawn from B.

7. The product of a few primes is ten times as much as the sum of the primes. What are these (not necessarily distinct) primes?

8. Four balls in space have radii 2, 2, 3 and 3 respectively. Each ball is tangent to the other three. There is another small ball inside the space between these four balls which is tangent to each of them.
What is the radius of this small ball?

9. Find the value of the expression ((…(((2*3)*4)*5)…)*2009)
where $x*y = \frac{x+y}{1+xy}$ for all positive x,y.

10. Let ABC be an acute angled triangle. Suppose that the altitude of triangle ABC at B intersects the circle with diameter AC at P and Q and the altitude at C intersects the circle with diameter AB at M and N.
Prove that P, Q, M and N lie on the circle.